Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 3

Specimen paper

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **both** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

Two IB schools, A and B, follow the IB Diploma Programme but have different teaching methods. A research group tested whether the different teaching methods lead to a similar final result.

For the test, a group of eight students were randomly selected from each school. Both samples were given a standardized test at the start of the course and a prediction for total IB points was made based on that test; this was then compared to their points total at the end of the course.

Previous results indicate that both the predictions from the standardized tests and the final IB points can be modelled by a normal distribution.

It can be assumed that:
- the standardized test is a valid method for predicting the final IB points
- that variations from the prediction can be explained through the circumstances of the student or school.

(a) Identify a test that might have been used to verify the null hypothesis that the predictions from the standardized test can be modelled by a normal distribution. [1]

(b) State why comparing only the final IB points of the students from the two schools would not be a valid test for the effectiveness of the two different teaching methods. [1]

(This question continues on the following page)

(Question 1 continued)

The data for school A is shown in the following table.

School A

Student number	Gender	Predicted IB points (p)	Final IB points (f)
1	male	43.2	44
2	male	36.5	34
3	female	37.1	38
4	male	30.9	28
5	male	41.1	39
6	female	35.1	39
7	male	36.4	40
8	male	38.2	38
	Mean	37.31	37.5

(c) For each student, the change from the predicted points to the final points ($f - p$) was calculated.

(i) Find the mean change.

(ii) Find the standard deviation of the changes. [3]

(d) Use a paired t-test to determine whether there is significant evidence that the students in school A have improved their IB points since the start of the course. [4]

(This question continues on the following page)

(Question 1 continued)

The data for school B is shown in the following table.

School B

Student number	Gender	Final IB points– Predicted IB points ($f-p$)
1	male	8.7
2	female	−1.1
3	female	4.8
4	female	−1.5
5	male	2.5
6	female	3.2
7	female	−1.3
8	female	3.1
	Mean	2.3

(e) (i) Use an appropriate test to determine whether there is evidence, at the 5% significance level, that the students in school B have improved more than those in school A.

(ii) State why it was important to test that both sets of points were normally distributed. [6]

(This question continues on the following page)

(Question 1 continued)

School A also gives each student a score for effort in each subject. This effort score is based on a scale of 1 to 5 where 5 is regarded as outstanding effort.

Student number	Gender	Predicted IB points	Final IB points	Average effort score
1	male	43.2	44	4.4
2	male	36.5	34	4.2
3	female	37.1	38	4.7
4	male	30.9	28	4.3
5	male	41.1	39	3.9
6	female	35.1	39	4.9
7	male	36.4	40	4.9
8	male	38.2	38	4.3
	Mean	37.31	37.5	4.45

It is claimed that the effort put in by a student is an important factor in improving upon their predicted IB points.

(f) (i) Perform a test on the data from school A to show it is reasonable to assume a linear relationship between effort scores and improvements in IB points. You may assume effort scores follow a normal distribution.

(ii) Hence, find the expected improvement between predicted and final points for an increase of one unit in effort grades, giving your answer to one decimal place. [4]

A mathematics teacher in school A claims that the comparison between the two schools is not valid because the sample for school B contained mainly girls and that for school A, mainly boys. She believes that girls are likely to show a greater improvement from their predicted points to their final points.

She collects more data from other schools, asking them to class their results into four categories as shown in the following table.

	$(f-p)<-2$	$-2\le(f-p)<0$	$0\le(f-p)<2$	$(f-p)\ge 2$
Male	6	8	10	9
Female	3	8	14	8

(g) Use an appropriate test to determine whether showing an improvement is independent of gender. [6]

(h) If you were to repeat the test performed in part (e) intending to compare the quality of the teaching between the two schools, suggest **two** ways in which you might choose your sample to improve the validity of the test. [2]

2. [Maximum mark: 28]

The number of brown squirrels, x, in an area of woodland can be modelled by the following differential equation.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1000}(2000 - x)\text{, where } x > 0$$

(a) (i) Find the equilibrium population of brown squirrels suggested by this model.

(ii) Explain why the population of squirrels is increasing for values of x less than this value. [3]

One year conservationists notice that some black squirrels are moving into the woodland. The two species of squirrel are in competition for the same food supplies. Let y be the number of black squirrels in the woodland.

Conservationists wish to predict the likely future populations of the two species of squirrels. Research from other areas indicates that when the two populations come into contact the growth can be modelled by the following differential equations, in which t is measured in tens of years.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1000}(2000 - x - 2y),\ x,\ y \geq 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{1000}(3000 - 3x - y),\ x,\ y \geq 0$$

An equilibrium point for the populations occurs when both $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$.

(b) (i) Verify that $x = 800$, $y = 600$ is an equilibrium point.

(ii) Find the other three equilibrium points. [6]

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(Question 2 continued)

When the two populations are small the model can be reduced to the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3y\,.$$

(c) (i) By using separation of variables, show that the general solution of $\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$ is $x = A\mathrm{e}^{2t}$.

(ii) Write down the general solution of $\frac{\mathrm{d}y}{\mathrm{d}t} = 3y$.

(iii) If both populations contain 10 squirrels at $t = 0$ use the solutions to parts (c) (i) and (ii) to estimate the number of black and brown squirrels when $t = 0.2$. Give your answers to the nearest whole numbers. [7]

For larger populations, the conservationists decide to use Euler's method to find the long-term outcomes for the populations. They will use Euler's method with a step length of 2 years ($t = 0.2$).

(d) (i) Write down the expressions for x_{n+1} and y_{n+1} that the conservationists will use.

(ii) Given that the initial populations are $x = 100$, $y = 100$, find the populations of each species of squirrel when $t = 1$.

(iii) Use further iterations of Euler's method to find the long-term population for each species of squirrel from these initial values.

(iv) Use the same method to find the long-term populations of squirrels when the initial populations are $x = 400$, $y = 100$. [7]

(e) Use Euler's method with step length 0.2 to sketch, on the same axes, the approximate trajectories for the populations with the following initial populations.

(i) $x = 1000$, $y = 1500$

(ii) $x = 1500$, $y = 1000$ [3]

(f) Given that the equilibrium point at $(800, 600)$ is a saddle point, sketch the phase portrait for $x \geq 0$, $y \geq 0$ on the same axes used in part (e). [2]